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Generalized Force in Classical Field Theory

J. KRAUSE

Departamento de Física Aplicada, Facultad de Ingeniería, Universidad Central de Venezuela, Caracas

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Abstract

The source strengths of the Euler-Lagrange equations, for a system of interacting fields, are heuristically interpreted as generalized forces. The canonical form of the energy-momentum tensor thus consistently appears, without recourse to space-time symmetry arguments. A concept of "conservative" generalized force in classical field theory is also briefly discussed.

1. *Introduction*

The purpose of this note is to present a simple derivation of the canonical form of the energy-momentum tensor in the classical theory of fields, without recourse to symmetry arguments à la Noether (1918).[1] Rather, as a guiding principle adopted to this end, one establishes an heuristic analogy with ordinary analytical mechanics. For the sake of concreteness, we present the issue for Lorentz covariant field theory,[2] albeit such restriction is not necessary for the following formalism to hold.

2. *The Source Strength as Generalized Force*

In relativistic linear field theory one usually describes a system of two interacting fields, say $Q^A(x)$ and $q^a(x)$, with $A = 1, \ldots, N$, $a = 1, \ldots, n$, and where $x = (x^\mu) = (x^0, x^1, x^2, x^3)$, by adopting a total Lagrangian density of the form

$$L(Q^A; q^a; Q^A_{,\mu}; q^a_{,\mu}) = L_Q(Q^A; Q^A_{,\mu}) + L_q(q^a; q^a_{,\mu}) + L_{Qq}(Q^A; Q^A_{,\mu}; q^a; q^a_{,\mu}) \tag{2.1}$$

[1] A brief historical review on Noether's theorem can be found in Schröder (1968); a simplified account is presented by Hill (1951). See also Rosen (1971). On the inversion of the theorem, see Candotti et al. (1970, 1972). An interesting generalization of Noether's theorem has been recently published by Rosen (1974a, 1974b).

[2] See, for instance, Bogoulioubov & Chirkov (1960), Chap. I.

where, clearly, L_Q and L_q are free field Lagrangians, and L_{Qq} denotes the interaction Lagrangian. The corresponding Euler-Lagrange field equations are of the form[3]

$$\partial L_Q/\partial Q^A - (D/Dx^\mu)(\partial L_Q/\partial Q^A_{,\mu}) = -F_A(Q^A; Q^A_{,\mu}; q^a; q^a_{,\mu}) \quad (2.2a)$$

$$\partial L_q/\partial q^a - (D/Dx^\mu)(\partial L_q/\partial q^a_{,\mu}) = -F_a(q^a; q^a_{,\mu}; Q^A; Q^A_{,\mu}) \quad (2.2b)$$

These equations are coupled through the inhomogeneity terms containing the sources of the fields, which are given by

$$F_A = \partial L_{Qq}/\partial Q^A - (D/Dx^\mu)(\partial L_{Qq}/\partial Q^A_{,\mu}) \quad (2.3a)$$

$$F_a = \partial L_{Qq}/\partial q^a - (D/Dx^\mu)(\partial L_{Qq}/\partial q^a_{,\mu}) \quad (2.3b)$$

As is well known, in analytical mechanics one introduces a Lagrangian $L(q^i, \dot{q}^i, t) = T(q^i, \dot{q}^i) - U(q^i, \dot{q}^i, t)$, describing a system with f degrees of freedom, $i = 1, \ldots, f$, where the kinetic energy term T corresponds to the free Lagrangian, $L_q = T$, while the generalized potential U gives us the interaction Lagrangian, i.e., $L_{qU} = -U$. The equations of motion for such a system are

$$\partial T/\partial q^i - (d/dt)(\partial T/\partial \dot{q}^i) = \partial U/\partial q^i - (d/dt)(\partial U/\partial \dot{q}^i) = -F_i$$

where the F_i's are the generalized forces. Accordingly, in the analytical theory of fields we interpret the source strengths F_A and F_a as generalized force densities; namely F_A corresponds to the generalized force (density) acting locally on the field $Q^A(x)$, due to the presence of the field $q^a(x)$ in the neighborhood of of the event x. A similar statement holds for F_a, *mutatis mutandi.* The Q field evolves under the influence of the force density F_A in such a manner that, as it changes from $Q^A(x)$ to $Q^A(x + dx) = Q^A(x) + dQ^A(x)$, say, a "work" is performed locally on the field according to the following definition (whose pattern we borrow from analytical mechanics):

$$dw_Q = F_A\, dQ^A = F_A Q^A_{,\mu}\, dx^\mu \quad (2.4a)$$

In the same manner, we also write

$$dw_q = F_a\, dq^a = F_a q^a_{,\mu}\, dx^\mu \quad (2.4b)$$

for the "work" performed on the q field. It is clear that the main difference between the concept of "work" in particle dynamics and in field theory stems from the fact that in the discrete case the displacement dx^μ is tangent to the particle's world-line, while in the continuous case dx^μ is a completely arbitrary world-displacement. Hence the path integrated "work" has no physical meaning in field theory. Nevertheless, the differentials dw have a simple local meaning. We will come back to this point presently.

[3] In this note we are using the concept of partial derivative with respect to the independent variables x^μ when the state functions and their gradients have been substituted as functions of the independent variables. Following Hill (1951), we use the symbol D/Dx^μ to denote the partial derivatives defined in this sense.

Let us now introduce the four-fource densities acting on the fields. Following the pattern of analytical mechanics, these we define as

$$f_\mu(Q) = F_A Q^A_{,\mu} \tag{2.5a}$$

$$f_\mu(q) = F_a q^a_{,\mu} \tag{2.5b}$$

Next we consider the total four-force (on the Q field) developed by the interaction, over a simple connected region R of space-time, namely,

$$F_\mu(Q) = \int_R d^4x f_\mu(Q) = \int_R d^4x F_A Q^A_{,\mu} \tag{2.6}$$

It is clear that, in order for the proposed interpretation to be valid at all, the presence of this four-force must be related to the total change in linear four-momentum pertaining to the Q field over the region R. This is, indeed, the case, for using equations (2.1) and (2.2) we have, after some manipulations,[4]

$$\begin{aligned} F_\mu(Q) &= \int_R d^4x[\partial L_{Qq}/\partial Q^A - (D/Dx^\nu)(\partial L_{Qq}/\partial Q^A_{,\nu})]Q^A_{,\mu} \\ &\stackrel{\circ}{=} -\int_R d^4x[\partial L_Q/\partial Q^A - (D/Dx^\nu)(\partial L_Q/\partial Q^A_{,\nu})]Q^A_{,\mu} \\ &= \int_R d^4x(D/Dx^\nu)[(\partial L_Q/\partial Q^A_{,\nu})Q^A_{,\mu} - \delta_\mu{}^\nu L_Q] \\ &= \oint_{\delta R} d\sigma_\nu[(\partial L_Q/\partial Q^A_{,\nu})Q^A_{,\mu} - \delta_\mu{}^\nu L_Q] \end{aligned} \tag{2.7}$$

where δR denotes the boundary of R. So we get, *on the orbit*,

$$F_\mu(Q) \stackrel{\circ}{=} \int_R d^4x T^\nu_{\mu,\nu}(Q) = \int_{\delta R} d\sigma_\nu T_\mu{}^\nu(Q) \tag{2.8}$$

with

$$T^\nu_\mu(Q) = (\partial L_Q/\partial Q^A_{,\nu})Q^A_{,\mu} - \delta^\nu_\mu L_Q \tag{2.9}$$

as it should be. Equation (2.8) represents the global force equation, while from (2.6) and (2.8) the local force equation, i.e., the four-force density equation, readily obtains

$$T^\nu_{\mu,\nu}(Q) \stackrel{\circ}{=} f_\mu(Q) \tag{2.10}$$

The Lagrangian formalism has a well-known gauge freedom, since the Lagrangian density used in the variational integral which will lead to a given set of Euler–Lagrange field equations is not unique. In effect, the most general gauge transformations of the Lagrangian density which leave the field equations form invariant correspond (of necessity and sufficiently) to[5]

$$L' = L + (D/Dx^\mu)G^\mu(Q^A; q^a) \tag{2.11}$$

[4] The symbol $\stackrel{\circ}{=}$ is used for an equality holding "along the orbit," i.e., once the Euler–Lagrange field equations have been used in order to obtain that equality; thus, the symbol $\stackrel{\circ}{=}$ denotes what is generally called a "weak" equality. See Candotti et al. (1970).

[5] Courant and Hilbert (1953), Vol. 1, pp. 193–196. See also Hill (1951).

where L is given in (2.1), say.[6] Since the arbitrary gauge functions G^μ do not depend on the gradient of the fields, we easily conclude that the generalized forces presented in equations (2.3), as well as the corresponding four-force densities defined in equations (2.5), are gauge invariant quantities under (2.11). In particular, we observe that the four-force density equation (2.10) is gauge invariant, notwithstanding the fact that the canonical tensor $T_\mu{}^\nu$ is a gauge-dependent object. We wish to remark this point here, for it seems that gauge invariance under (2.11) is a very natural requirement for having a well-defined physically meaningful object. In this sense, we also wish to remark that the canonical $T_\mu{}^\nu$ tensor participates in the dynamics (via Gauss' theorem) only through its divergence $T^\nu_{\mu,\nu}$, i.e., a gauge invariant quantity, as it should be.

3. "*Conservative*" *Generalized Forces*

Using the same kind of arguments followed in the previous section, it is easy to prove that for the total system we have

$$T_\mu{}^\nu(Q;q) = T_\mu{}^\nu(Q) + T_\mu{}^\nu(q) + (\partial L_{Qq}/\partial Q^A_{,\nu})Q^A_{,\mu} + (\partial LQq/\partial q^a_{,\nu})q^a_{,\mu} - \delta_\mu{}^\nu L_{Qq} \tag{3.1}$$

where now the total energy-momentum conservation law holds, i.e.,

$$T^\nu_{\mu,\nu}(Q;q) = 0 \tag{3.2}$$

Hence we obtain the local force equation for the total (closed) system in the form

$$F_A Q^A_{,\mu} + F_a Q^a_{,\mu} + (D/Dx^\nu)\{(\partial L_{Qq}/\partial Q^A_{,\nu})Q^A_{,\mu} + (\partial L_{Qq}/\partial q^a_{,\nu})q^a_{,\mu} - \delta_\mu{}^\nu L_{Qq}\} = 0 \tag{3.3}$$

and therefore the following "energy" density relation holds locally:

$$dw_Q + dw_q + dU_{Qq} = dW_{Qq} \tag{3.4}$$

Here we have defined, in the usual way,

$$dU_{Qq} = -dL_{Qq} \tag{3.5}$$

as the (analog of the) change in "potential energy" density of the system, and

$$dW_{Qq} = -dx^\mu(D/Dx^\nu)[(\partial L_{Qq}/\partial Q^A_{,\nu})Q^A_{,\mu} + (\partial L_{Qq}/\partial q^a_{,\nu})q^a_{,\mu}] \tag{3.6}$$

Thus, the necessary and sufficient condition for the "energy" density conservation law to hold locally, namely, to have

$$dw_Q + dw_q + dU_{Qq} = 0 \tag{3.7}$$

[6] To be sure, in equation (2.11) we have omitted an arbitrary multiplicative constant on L, for such scale transformations of the Lagrangian density bear no interest in the present work.

is that the expression in equation (3.6) vanish identically for arbitrary space-time displacements, i.e.,

$$(D/Dx^{\nu})[(\partial L_{Qq}/\partial Q^A_{,\nu})Q^A_{,\mu} + (\partial L_{Qq}/\partial q^a_{,\nu})q^a_{,\mu}] = 0 \tag{3.8}$$

Here we face a situation that is quite reminiscent of what we know from ordinary analytical mechanics concerning nonconservative forces derivable from a generalized potential. For instance, let us formally consider a two-particle system represented by the Lagrangian

$$L = (m/2)\dot{\mathrm{x}}_1{}^2 + (m'/2)\dot{\mathrm{x}}_2{}^2 - U(\mathrm{x}_1, \dot{\mathrm{x}}_1, \mathrm{x}_2, \dot{\mathrm{x}}_2) \tag{3.9}$$

For such a system, the following energy relation readily obtains:

$$dw_1 + dw_2 + dU = dW = dt\{[(d/dt)(\partial U/\partial \dot{\mathrm{x}}_1)]\cdot\dot{\mathrm{x}}_1 + [(d/dt)(\partial U/\partial \dot{\mathrm{x}}_2)]\cdot\dot{\mathrm{x}}_2\} \tag{3.10}$$

where, clearly, the dw's are $dw = d(m\dot{\mathrm{x}}^2/2)$. Hence, the interaction is not a conservative one, unless the expression for dW vanishes for arbitrary dt. Accordingly, in classical field theory we may state that equation (3.8) represents a necessary and sufficient condition for the generalized forces (due to the L_{Qq} interaction) to be "conservative" forces, in the sense that a conservation law for the "energy" density of the total system holds locally, Obviously, we cannot claim a clear and unique physical meaning for the dW_{Qq} term in classical field theory, as there is no clear unique meaning for the dW term in ordinary analytical mechanics. Clearly so, for these terms depend quite generally on the kind of nonconservative interaction involved.

There is an important difference, however, between both contexts, for in analytical mechanics a discrete system behaving as in equations (3.9) and (3.10) is usually regarded as an open system (i.e., an incomplete description of a closed one). In this manner we rescue the physical law of (total) energy conservation. On the other hand, in the heuristic approach to classical field theory a system behaving as in (2.1) and (3.4) is nevertheless regarded as closed (i.e., a complete system). In effect, the conservation law of real physical interest for field theory is stated in equation (3.2). Thus, we do not claim the heuristic analogy we are discussing in this paper to be *perfect.* Due caution is needed, of course, in order to explore field theory while adopting the formal "mechanistic" standpoint.[7]

Finally, we comment that a glance at equation (3.4) may tempt us to think of the dW_{Qq} term as the field's "heat" produced by the interaction, and since a field theoretic concept of "heat" is clearly untenable one would conclude that

[7] For instance, one may tickle the argument and observe that, *sensu stricto*, according to equation (3.6), the analytical mechanic analog to dW_{Qq} would be formally given by $dW' = dt(d/dt)[(\partial U/\partial\dot{\mathrm{x}}_1)\cdot\dot{\mathrm{x}}_1 + (\partial U/\partial\dot{\mathrm{x}}_2)\cdot\dot{\mathrm{x}}_2]$, and not by dW, for t is the independent variable in analytical mechanics, as are the x^{μ}'s in field theory. It seems that one should not push the analogy that far, since according to this $t \leftrightarrow x^{\mu}$ argument, the analog to the total $T_{\mu}{}^{\nu}(Q;q)$ tensor would be the total Hamiltonian $H = T_1 + T_2 + U$. But then, the analog to equation (3.2) would be $dH/dt = 0$, which does not hold quite generally in analytical mechanics; while (3.2) is quite general in classical field theory.

dW_{Qq} must be an exact differential.[8] It must be borne in mind, however, as we have already remarked in Section 2, that the path integrated "work" has no cogent physical meaning in the theory of fields and, hence, there is no need to impose integrability conditions on dW_{Qq}.

4. *Conclusion*

We observe that these field forces are not due to a direct action of one field on the other, for such "contact" forces would require $dw_Q + dw_q = 0$, which is not the case if there is an interaction at all. These forces manifest themselves locally through the *interaction compartment*, represented by the term L_{Qq} in the total Lagrangian, in much the same manner as the interaction forces in a system of, say, two particles connected by a spring. Thus, for instance, if for some dx^μ we have $dw_Q > 0$, we interpret this as a *local transfer of energy density* from the interaction compartment to the Q field.

We conclude that the interpretation of the source strengths as generalized forces is consistent with the well-known dynamical meaning of $T_\mu{}^\nu(Q)$ as the energy-momentum transport tensor of the Q field. In this way one obtains the usual canonical form of this tensor without recourse to Noether's theorem. Finally, while from the standpoint of the canonical theory of fields, Noether's (first) theorem clearly plays a very central role in the formalism, we wish to remark, however, that once the canonical form of the $T_\mu{}^\nu$ tensor has been obtained along the usual space-time symmetry arguments, it is interesting to be able to grasp its dynamical meaning directly by analogy with ordinary analytical mechanics.

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[8] As a matter of fact, most interaction Lagrangians usually introduced in applied field theory met this requirement. See, for instance, Bogoulioubov & Chirkov (1960), p. 70.